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COSINE AND SINE (CAS) WAVELET COLLOCATION METHOD FOR THE NUMERICAL SOLUTION OF INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT
Cosine and Sine (CAS) wavelet collocation method for the numerical solution of Volterra, Fredholm integral and integro-differential equations, mixed Volterra-Fredholm integral equations. The method is based Cosine and Sine (CAS) wavelet approximations. The Cosine and Sine (CAS) wavelet is first presented and the resulting Cosine and sine wavelet matrices are utilized to reduce the integral and integro-differential equations into a system of algebraic equations, which is the required Cosine and Sine (CAS) coefficients, are computed using Matlab. The technique is tested on some numerical examples and compared with the exact and existing methods (i.e., Hermite, Legendre and Bernoulli Wavelet). Error analysis is worked out, which shows efficiency of the proposed scheme.

KEYWORDS: Cosine and Sine (CAS) wavelet, Collocation method, Integral equations, Integro-differential equations.

I. INTRODUCTION
Integral and integro-differential equations found its applications in several fields of science and engineering. There are some numerical methods for approximating the solution of integral and integro-differential equations of second kind are known and many basis functions have been used [1]. Delves and Mohammed [2] have introduced the computational methods for solving integral equations. In recent years, wavelets have established many different fields of science and engineering. Applications of different wavelets have been introduced for solving integral and integro-differential equations. Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [3, 4]. Since 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations, a detailed survey on these papers can be found in [5]. Namely, Lepik and Tamme [6-11] applied the Haar wavelet method. Maleknejad et al. [12-16] has introduced rationalized haar wavelets, Legendre wavelets, Hermite cubic spline wavelet, Coifman wavelet. Babolian et. al [17] have applied chebyshev wavelet operational matrix of integration. Galerkin method for the constructions of orthonormal wavelet bases approached by Liang et.al [18]. Yousefi et al. [19] have introduced a new cosine and sine wavelet. Shiralashetti and Mundewadi [24] introduced Bernoulli wavelet method for solving Fredholm integral equations. In this paper, we introduced a new approach for solving integral and integro-differential equations using cosine and sine (CAS) wavelet.

II. PROPERTIES OF COSINE AND SINE (CAS) WAVELETS
Wavelets
Recently, wavelets based numerical methods applied extensively for signal processing and physics research has proved to be an amazing mathematical tool. Wavelets can be used for algebraic manipulations in the system of equations obtained better resulting system. Wavelets constitute a family of functions constructed from dilation

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and translation of a single function called the mother wavelet. When the dilation parameter 'a' and the translation parameter 'b' vary continuously, we have the following family of continuous wavelets [19];

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in R, \quad a \neq 0 \tag{2.1}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, b = p b_0 a_0^{-k}, a_0 > 1, b_0 > 0$ and p, and k positive integer, from Eq. (2.1) we have the following family of discrete wavelets:

$$\psi_{k,p}(t) = |a|^{-\frac{1}{2}} \psi(a_0^k t - p b_0)$$

where $\psi_{k,p}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,p}(t)$ forms an orthonormal basis.

Cosine and Sine (CAS) Wavelet

Cosine and Sine wavelet $C_{n,m}(t) = C(k, n, m, t)$ have four arguments; $k = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots, 2^k$, $m = 0, 1, \ldots, M-1$ and t is the normalized time.

For any positive integer k, the cosine and sine wavelets family is defined in the interval [0, 1) as follows;

$$C_{n,m}(t) = \begin{cases} 2^{k/2} CAS_m(2^k t - n), & for \quad \frac{n-1}{2^k} \le t < \frac{n}{2^k} \\ 0, & Otherwise \end{cases} \tag{2.2}$$

where $CAS_m(t) = \cos(2m\pi t) + \sin(2m\pi t)$.

Equivalently, by computational procedure for any positive integer k, the CAS wavelets family is defined as follows;

$$C_i(t) = \begin{cases} 2^{k/2} CAS_m(2^k t - n), & for \quad \frac{n-1}{2^k} \le t < \frac{n}{2^k} \\ 0, & Otherwise \end{cases} \tag{2.3}$$

where $i = n + 2^k m$. By varying the values of i with respect to the collocation points $t_j = \frac{j-0.5}{N}$, $j = 1, 2, \ldots, N$, we get the CAS wavelet matrix of order $N \times N$, where $N = 2^k M$.

For k = 1 implies n = 1, 2 and M = 3 implies m = 0, 1, 2 then Eq. (2.2) gives the CAS wavelet matrix of order ($N = 2^k M$) 6x6 as,

$$C_{6\times 6} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ 1.9319 & -1.4142 & -0.5176 & 0 & 0 & 0 \\ 0.5176 & 1.4142 & -1.9319 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 1.9319 & -1.4142 & -0.5176 \\ 0 & 0 & 0 & 0.5176 & 1.4142 & -1.9319 \end{bmatrix}$$

For k = 1 and M = 4 of order 8x8 as,

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$$C_{8\times 8}=\begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 & 0 \\ 2.0000 & -0.0000 & -2.0000 & 0.0000 & 0 & 0 & 0 & 0 \\ 1.4142 & -1.4142 & 1.4142 & -1.4142 & 0 & 0 & 0 & 0 \\ 0.0000 & 2.0000 & 0.0000 & -2.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0 & 2.0000 & -0.0000 & -2.0000 & 0.0000 \\ 0 & 0 & 0 & 0 & 1.4142 & -1.4142 & 1.4142 & -1.4142 \\ 0 & 0 & 0 & 0 & 0.0000 & 2.0000 & 0.0000 & -2.0000 \end{bmatrix}$$

III. COSINE AND SINE (CAS) WAVELET COLLOCATION METHOD OF SOLUTION

In this section, we present a Cosine and sine wavelet (CAS) collocation method for solving integral and integro-differential equations,

Integral Equations

Fredholm Integral equations

Consider the Fredholm integral equations,

$$u(t)=f(t)+\int_0^1 k_1(t,s)u(s)\,ds, \tag{3.1}$$

where $f(t)\in L^2[0,1)$, $k_1(t,s)\in L^2([0,1)\times[0,1))$ and $u(t)$ is an unknown function.

Let us approximate $f(t), u(t)$, and $k_1(t,s)$ by using the collocation points t_i as given in the above section 2.2. Then the numerical procedure as follows:

STEP 1: Let us first approximate $f(t) \approx X^T\Psi(t)$ and $u(t) \approx Y^T\Psi(t)$, (3.2)

Let the function $f(t)\in L^2[0,1]$ may be expanded as:

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} x_{n,m}C_{n,m}(t), \tag{3.3}$$

where

$$x_{n,m}=(f(t),C_{n,m}(t)). \tag{3.4}$$

In (3.4), (. , .) denotes the inner product.

If the infinite series in (3.3) is truncated, then (3.3) can be rewritten as:

$$f(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} x_{n,m}C_{n,m}(t)=X^T\Psi(t), \tag{3.5}$$

where X and $\Psi(t)$ are $N\times 1$ matrices given by:

$$\begin{aligned} X &=[x_{10},x_{11},\ldots,x_{1,M-1},x_{20},\ldots,x_{2,M-1},\ldots,x_{2^{k-1},0},\ldots,x_{2^{k-1},M-1}]^T \\ &=[x_1,x_2,\ldots,x_{2^{k-1}M}]^T, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \Psi(t)&=[C_{10}(t),C_{11}(t),\ldots,C_{1,M-1}(t),C_{20}(t),\ldots,C_{2,M-1}(t),\ldots,C_{2^{k-1},0}(t),\ldots,C_{2^{k-1},M-1}(t)]^T \\ &=[C_1(t),C_2(t),\ldots,C_{2^{k-1}M}(t)]^T. \end{aligned} \tag{3.7}$$

STEP 2: Next, approximate the kernel function as: $k_1(t,s)\in L^2([0,1]\times[0,1])$

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$$k_1(t,s) \approx \Psi^T(t) K_1 \Psi(s), \tag{3.8}$$

where K_1 is $2^k M \times 2^k M$ matrix, with

$$[K_1]_{ij} = (C_i(t), (k_1(t,s), C_j(s))).$$

$$\text{i.e., } K_1 \approx \left[\Psi^T(t)\right]^{-1} \cdot \left[k_1(t,s)\right] \cdot \left[\Psi(s)\right]^{-1} \tag{3.9}$$

STEP 3: Substituting Eq. (3.2) and Eq. (3.8) in Eq. (3.1), we have:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t) K_1 \Psi(s) \Psi^T(s) Y ds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t) K_1 \left(\int_0^1 \Psi(s) \Psi^T(s)\, ds \right) Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_1 Y),$$

Then we get a system of equations as,

$$(I - K_1)Y = X. \tag{3.10}$$

By solving this system obtain the vector CAS wavelet coefficients '*Y*' and substituting in step 4.

STEP 4: $u(t) \approx Y^T \Psi(t)$

This is the required approximate solution of Eq. (3.1).

Volterra Integral equations

Consider the Volterra integral equations with convolution but non-symmetrical kernel

$$u(t) = f(t) + \int_0^t k_2(t,s) u(s)\, ds, \quad t \in [0,1] \tag{3.11}$$

where $f(t) \in L^2[0,1)$, $k_2(t,s) \in L^2([0,1) \times [0,1))$ and $u(t)$ is an unknown function.

Let us approximate $f(t), u(t)$, and $k_2(t,s)$ by using the collocation points t_i as given in the above section 2.2. Then the numerical procedure as follows:

STEP 1: The Eq. (3.11) can be rewritten in Fredholm integral equations, with a modified kernel $\tilde{k}_2(t,s)$ and solved in Fredholm form [20] as,

$$u(t) = f(t) + \int_0^t \tilde{k}_2(t,s) u(s)\, ds, \tag{3.12}$$

where, $\tilde{k}_2(t,s) = \begin{cases} k_2(t,s), & 0 \le s \le t \\ 0, & t \le s \le 1. \end{cases}$

STEP 2: Let us first approximate $f(t)$ and $u(t)$ as given in Eq. (3.2),

STEP 3: Next, we approximate the kernel function as: $\tilde{k}_2(t,s) \in L^2([0,1] \times [0,1])$

$$\tilde{k}_2(t,s) \approx \Psi^T(t) \cdot K_2 \cdot \Psi(s), \tag{3.13}$$

where K_2 is $2^k M \times 2^k M$ matrix, with

$$(K_2)_{ij} = (C_i(t), (\tilde{k}_2(t,s), C_j(s))).$$

$$\text{i.e., } K_2 \approx \left[\Psi^T(t)\right]^{-1} \cdot \left[\tilde{k}_2(t,s)\right] \cdot \left[\Psi(s)\right]^{-1} \tag{3.14}$$

STEP 4: Substituting Eq. (3.2) and Eq. (3.13) in Eq. (3.12), we have:

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$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t)K_2\Psi(s)\Psi^T(s)Yds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t)K_2\left(\int_0^1 \Psi(s)\Psi^T(s)\,ds\right)Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_2Y),$$

Then we get a system of equations as,

$$(I - K_2)Y = X. \tag{3.15}$$

By solving this system obtain the vector CAS wavelet coefficients '*Y*' and substituting in step 5.

STEP 5: $u(t) \approx Y^T\Psi(t)$

This is the required approximate solution of Eq. (3.11).

Fredholm-Volterra integral equations

Consider the Fredholm-Volterra integral equation of the second kind,

$$u(t) = f(t) + \int_0^1 k_1(t,s)u(s)\,ds + \int_0^t k_2(t,s)u(s)\,ds, \tag{3.16}$$

where $f \in L^2[0,1),\ k_1 \text{ and } k_2 \in L^2([0,1)\times[0,1))$ are known function and $u(t)$ is an unknown function.

Let us approximate $f(t), u(t), k_1(t,s)$ and $k_2(t,s)$ by using the collocation points as follows:

STEP 1: Let us first approximate $f(t)$ and $u(t)$ as given in Eq. (3.2),

STEP 2: Substituting Eq. (3.2), Eq. (3.9) and Eq. (3.14) in Eq. (3.16), we get a system of *N* equations with *N* unknowns,

$$\text{i.e., } (I - K_1 - K_2)Y = X. \tag{3.17}$$

where, I is an identity matrix.

By solving this system we obtain the CAS wavelet coefficient '*Y*' and substituting '*Y*' in step 3.

STEP 3: $u(t) \approx Y^T\Psi(t)$

This is the required approximate solution of Eq. (3.16).

Integro-differential Equations

Fredholm Integro-differential equations

In this section, we concerned about a technique that will reduce Fredholm integro-differential equation to an equivalent Fredholm integral equation. This can be easily done by integrating both sides of the integro-differential equation as many times as the order of the derivative involved in the equation from 0 to *t* for every time we integrate, and using the given initial conditions. It is worth noting that this method is applicable only if the Fredholm integro-differential equation involves the unknown function *u*(*t*) only, and not any of its derivatives, under the integral sign [1].

Consider the Fredholm integro-differential equations,

$$u^{(n)}(t) = f(t) + \int_0^1 k_1(t,s)u(s)\,ds,\ u^{(l)} = b_l, \tag{3.18}$$

where $f(t) \in L^2[0,1),\ k_1(t,s) \in L^2([0,1)\times[0,1))$ and $u^{(n)}(t)$ is an unknown function.

where $u^{(n)}(t)$ is the n^{th} derivative of $u(t)$ with respect to *t* and b_l are constants that define the initial conditions.

Let us first, we convert the Fredholm integro-differential equation into Fredholm integral equation, then we reduce it into a system of algebraic equations as given in Eq. (3.10), using this system we solve the Eq. (3.18).

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Volterra Integro-differential equations

In this section, we concerned with converting to Volterra integral equations. We can easily convert the Volterra integro-differential equation to equivalent Volterra integral equation, provided the kernel is a difference kernel defined by $k(t, s) = k(t - s)$. This can be easily done by integrating both sides of the equation and using the initial conditions. To perform the conversion to a regular Volterra integral equation, we should use the well-known formula, which converts multiple integrals into a single integral [1].

i.e.,

$$\int_0^t\int_0^t.......\int_0^t u(t)dt^n = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u(s)ds$$

Consider the Volterra integro-differential equations,

$$u^{(n)}(t) = f(t) + \int_0^t k_2(t,s)u(s)\,ds,\ u^{(l)} = b_l, \tag{3.19}$$

where $f(t) \in L^2[0,1)$, $k_2(t,s) \in L^2([0,1)\times[0,1))$ and $u^{(n)}(t)$ is an unknown function.

where $u^{(n)}(t)$ is the n^{th} derivative of $u(t)$ with respect to t and b_l are constants that define the initial conditions.

Let us first, we convert the Volterra integro-differential equation into Volterra integral equation, then we reduce it into a system of algebraic equations as given in Eq. (3.15), using this system we solve the Eq. (3.19).

IV. CONVERGENCE ANALYSIS

Theorem: The series solution $u(t) = \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} x_{p,q}C_{p,q}(t)$ defined in Eq. (3.5) using CAS wavelet method converges to $u(t)$ as given in [21].

Proof: Let $L^2(R)$ be the Hilbert space and $C_{p,q}$ defined in Eq. (3.2) forms an orthonormal basis.

Let $u(t) = \sum_{i=0}^{M-1} x_{p,i}C_{p,i}(t)$ where $x_{p,i} = \langle u(t), C_{p,i}(t)\rangle$ for a fixed p.

Let us denote $C_{p,i}(t) = C(t)$ and let $\alpha_j = \langle u(t), C(t)\rangle$.

Now we define the sequence of partial sums S_p of $(\alpha_j C(t_j))$; Let S_p and S_p be the partial sums with $p \geq q$.

We have to prove S_p is a Cauchy sequence in Hilbert space.

Let $S_p = \sum_{i=1}^{p} \alpha_j C(t_j)$.

$$\text{Now } \langle u(t), S_p\rangle = \left\langle u(t), \sum_{i=1}^{p}\alpha_j C(t_j)\right\rangle = \sum_{j=1}^{p}|\alpha_j|^2.$$

We claim that $\|S_p - S_p\|^2 = \sum_{j=q+1}^{p}|\alpha_j|^2,\ p > q.$

Now

$$\left\|\sum_{j=q+1}^{p}\alpha_j C(t_j)\right\|^2 = \left\langle \sum_{j=q+1}^{p}\alpha_j C(t_j), \sum_{j=q+1}^{p}\alpha_j C(t_j)\right\rangle = \sum_{j=q+1}^{p}|\alpha_j|^2,\ \ for\ \ p > q.$$

Therefore, $\left\|\sum_{j=q+1}^{p}\alpha_j C(t_j)\right\|^2 = \sum_{j=1}^{p}|\alpha_j|^2,\ \ for\ \ p > q.$

From Bessel's inequality, we have $\sum_{j=1}^{p}|\alpha_j|^2$ is convergent and hence

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$$\left\| \sum_{j=q+1}^{p} \alpha_j C(t_j) \right\|^2 \to 0 \quad as \quad q, p \to \infty$$

So, $\left\| \sum_{j=q+1}^{p} \alpha_j C(t_j) \right\| \to 0$ and $\{S_p\}$ is a Cauchy sequence and it converges to *s* (say).

We assert that $u(t) = s$.

Now $\langle s - u(t), C(t_j) \rangle = \langle s, C(t_j) \rangle - \langle u(t), C(t_j) \rangle = \left\langle \lim_{p\to\infty} S_p, C(t_j) \right\rangle - \alpha_j = \alpha_j - \alpha_j$,

This implies,

$$\langle s - u(t), C(t_j) \rangle = 0$$

Hence $u(t) = s$ and $\sum_{i=1}^{p} \alpha_j C(t_j)$. converges to $u(t)$ as $p \to \infty$ and proved.

V. NUMERICAL EXPERIMENTS

In this section, we present CAS wavelet collocation method for the numerical solution of integral and integro-differential equation in comparison with existing methods (i.e., Haar Wavelet (HW) [5], Legendre Wavelet (LW) [14], Bernoulli Wavelet (BW) [24]) to demonstrate the capability of the proposed method and error analysis are shown in tables and figures. Error function is presented to verify the accuracy and efficiency of the following numerical results:

$$E_{\max} = Error\ function = \| u_e(t_i) - u_a(t_i) \|_{\infty} = \sqrt{\sum_{i=1}^{n} \left(u_e(t_i) - u_a(t_i) \right)^2}$$

where u_e and u_a are the exact and approximate solution respectively.

Example 5.1 Let us consider the Fredholm integral equation of the second kind [13],

$$u(t) = t + \int_0^1 k(t,s) u(s)\, ds\,, \quad 0 \le t \le 1 \tag{5.1}$$

which has the exact solution $u(t) = \sec(1)\sin(t)$. Where $f(t) = t$ and $k_1(t,s) = \begin{cases} t, & t \le s \\ s, & s \le t. \end{cases}$

Firstly, we approximate $f(t) \simeq X^T \Psi(t)$, and $u(t) \simeq Y^T \Psi(t)$,

Next, approximate the kernel function as: $k_1(t,s) \in L^2([0,1]\times[0,1])$

$$k_1(t,s) \simeq \Psi^T(t) K_1 \Psi(s),$$

where K_1 is $2^k M \times 2^k M$ matrix, with $[K_1]_{ij} = (H_i(t), (k_1(t,s), H_j(s)))$.

$$K_1 \simeq \left[\Psi^T(t)\right]^{-1} \cdot [k_1(t,s)] \cdot [\Psi(s)]^{-1}$$

Next, substituting the function $f(t)$, $u(t)$, and $k_1(t,s)$ in Eq. (5.1), then using the collocation points, we get the system of algebraic equations with unknown coefficients for k = 1 and M = 4 (N = 8), as an order 8×8 as follows:

$$\Psi^T(t) Y = \Psi^T(t) X + \int_0^1 \Psi^T(t) K_1 \Psi(s) \Psi^T(s) Y ds$$

$$\Psi^T(t) Y = \Psi^T(t) X + \Psi^T(t) K_1 \left(\int_0^1 \Psi(s) \Psi^T(s)\, ds \right) Y$$

$$\Psi^T(t) Y = \Psi^T(t)(X + K_1 Y),$$

$$(I - K_1)Y = X, \text{ where, } I = \int_0^1 \Psi(s)\Psi^T(s)\,ds \text{ is the identity matrix.}$$

where, X = [0.1768 -0.0625 -0.0442 -0.0625 0.5303 -0.0625 -0.0442 -0.0625],

$$K_1 = \begin{bmatrix} 0.0859 & -0.0276 & -0.0156 & -0.0166 & 0.1250 & -0.0000 & 0.0000 & 0.0000 \\ -0.0276 & 0.0156 & 0.0055 & 0.0078 & -0.0442 & 0.0000 & -0.0000 & -0.0000 \\ -0.0156 & 0.0055 & 0.0078 & 0.0055 & -0.0312 & 0.0000 & -0.0000 & -0.0000 \\ -0.0166 & 0.0078 & 0.0055 & 0.0156 & -0.0442 & 0.0000 & -0.0000 & -0.0000 \\ 0.1250 & -0.0442 & -0.0312 & -0.0442 & 0.3359 & -0.0276 & -0.0156 & -0.0166 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0276 & 0.0156 & 0.0055 & 0.0078 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0156 & 0.0055 & 0.0078 & 0.0055 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0166 & 0.0078 & 0.0055 & 0.0156 \end{bmatrix}$$

By solving this system of equations, we obtain the CAS wavelet coefficients,
Y = [0.3212 -0.1135 -0.0787 -0.1100 0.8847 -0.0893 -0.0594 -0.0794]
and substituting these coefficients in $u(t) = Y^T\Psi(t)$, we get the approximate solution $u(t)$ are shown in table 1. Maximum Error analysis is shown in table 2 and compared with existing methods.

Table 1: Numerical results of the example 5.1

t	Exact	CAS Wavelet
0.0625	0.1156	0.1158
0.1875	0.3450	0.3456
0.3125	0.5690	0.5699
0.4375	0.7841	0.7854
0.5625	0.9870	0.9886
0.6875	1.1745	1.1764
0.8125	1.3437	1.3457
0.9375	1.4919	1.4941

Table 2: Maximum error analysis of the example 5.1

N	$E_{\max}$ (HW)	$E_{\max}$ (LW)	$E_{\max}$ (BW)	$E_{\max}$ (CAS)
8	5.15e-02	8.67e-03	4.34e-03	2.18e-03
16	1.39e-02	1.21e-03	1.07e-03	5.52e-04
32	3.59e-03	2.77e-04	2.68e-04	1.38e-04
64	9.07e-04	6.77e-05	6.72e-05	3.47e-05
128	2.27e-04	1.68e-05	1.68e-05	8.70e-06

Example 5.2 Next, consider [15],

$$u(t) = \sin(2\pi t) + \int_0^1 \cos(t)u(s)\,ds. \tag{5.2}$$

which has the exact solution $u(t) = \sin(2\pi t)$. We applied the CAS wavelet approach and solved Eq. (5.2), we get the CAS wavelet coefficients 'Y 'and substitute in $u(t)\,;\; Y^T\Psi(t),$ we obtain the approximate solution with exact solutions. Error analysis is compared with existing method is shown in table 3.

Example 5.3 Next, consider [15],

$$u(t) = \sin(2\pi t) + \int_0^1 (t^2 - t - s^2 + s)u(s)\,ds \tag{5.3}$$

which has the exact solution $u(t) = \sin(2\pi t)$. Solving Eq. (5.3), we get the CAS wavelet coefficients by using the present method and substitute in $u(t) ; Y^T\Psi(t),$ we obtain the approximate solution with exact solution. Error analysis is compared with existing method is shown in table 3.

Example 5.4 Next, consider [15],

$$u(t) = -2t^3 + 3t^2 - t + \int_0^1 (t^2 - t - s^2 + s)u(s)\,ds \tag{5.4}$$

solving Eq. (5.4) , we get the CAS wavelet coefficients by using the present method and substitute in $u(t) ; Y^T\Psi(t),$ we obtain the approximate solution with exact solutions $u(t) = -2t^3 + 3t^2 - t$. Error analysis is compared with existing method is shown in table 3.

Table 3: Comparison of the error analysis

N	Example 5.2		Example 5.3		Example 5.4	
	Method [15]	CAS Wavelet	Method [15]	CAS Wavelet	Method [15]	CAS Wavelet
4	2.84e-02	6.66e-16	2.84e-02	1.11e-16	1.33e-10	0
8	2.38e-03	4.99e-16	2.38e-03	3.33e-16	3.79e-10	2.77e-17
16	2.09e-04	4.44e-16	2.10e-04	4.44e-16	3.26e-10	4.16e-17
32	1.20e-04	7.77e-16	2.00e-04	5.55e-16	4.83e-10	4.16e-17

Example 5.5 Next, consider the Volterra integral equation of the second kind [20],

$$u(t) = \sin(t) + \int_0^t (t-s)u(s)\,ds, \qquad 0 \le t \le 1 \tag{5.5}$$

which has the exact solution $u(t) = \frac{1}{2}(\sin t + \sinh t).$ Where $f(t) = \sin(t)$ and kernel $k_2(t,s) = (t-s)$.

Firstly, we approximate $f(t) ; X^T\Psi(t),$ and $u(t) ; Y^T\Psi(t),$

Next, approximate the kernel function as: $k_2(t,s) \in L^2([0,1]\times[0,1])$

$$k_2(t,s) ; \Psi^T(t)K_2\Psi(s),$$

where K_2 is $2^k M \times 2^k M$ matrix, with $[K_2]_{ij} = (H_i(t),(k_2(t,s),H_j(s))).$

$$K_2 ; \left[\Psi^T(t)\right]^{-1}\cdot\left[k_2(t,s)\right]\cdot\left[\Psi(s)\right]^{-1}$$

Next, substituting the $f(t), u(t)$, and $k_2(t,s)$ in Eq. (5.5) using the collocation point, we get the system of algebraic equations with unknown coefficients for k = 1 and M = 4 (N = 8), as an order 8×8 as follows:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t)K_2\Psi(s)\Psi^T(s)Y ds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t)K_2\left(\int_0^1 \Psi(s)\Psi^T(s)\,ds\right)Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_2Y),$$

$$(I - K_2)Y = X, \quad \text{where, } I = \int_0^1 \Psi(s)\Psi^T(s)\,ds \text{ is the identity matrix.}$$

where, X = [0.1732 -0.0612 -0.0425 -0.0593 0.4773 -0.0482 -0.0321 -0.0429],

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$$K_2 = \begin{bmatrix} 0.0391 & 0.0276 & 0.0156 & 0.0166 & 0 & 0 & 0 & 0 \\ -0.0166 & -0.0156 & -0.0055 & -0.0078 & 0 & 0 & 0 & 0 \\ -0.0156 & -0.0055 & -0.0078 & -0.0055 & 0 & 0 & 0 & 0 \\ -0.0276 & -0.0078 & -0.0055 & -0.0156 & 0 & 0 & 0 & 0 \\ 0.2500 & 0.0442 & 0.0313 & 0.0442 & 0.0391 & 0.0276 & 0.0156 & 0.0166 \\ -0.0442 & -0.0000 & -0.0000 & -0.0000 & -0.0166 & -0.0156 & -0.0055 & -0.0078 \\ -0.0312 & 0.0000 & 0.0000 & 0.0000 & -0.0156 & -0.0055 & -0.0078 & -0.0055 \\ -0.0442 & 0.0000 & 0 & 0.0000 & -0.0276 & -0.0078 & -0.0055 & -0.0156 \end{bmatrix}$$

$Y =$ [0.1767 -0.0625 -0.0442 -0.0625 0.5318 -0.0631 -0.0448 -0.0636]

By solving this system of equations, we obtain the CAS wavelet coefficients 'Y' and substituting these coefficients in $u(t)$; $Y^T\Psi(t)$, we get the approximate solution $u(t)$ as shown in table 4. Maximum error analysis is compared with existing methods is shown in table 4.

Table 4: Numerical results of the example 5.5

t	Exact	CAS
0.0625	0.0625	0.0625
0.1875	0.1875	0.1874
0.3125	0.3125	0.3123
0.4375	0.4376	0.4373
0.5625	0.5630	0.5626
0.6875	0.6888	0.6883
0.8125	0.8155	0.8149
0.9375	0.9435	0.9428

Table 5: Maximum error analysis of the example 5.5

N	$E_{\max}$(HW)	$E_{\max}$(LW)	$E_{\max}$(BW)	$E_{\max}$(CAS)
8	2.42e-02	2.07e-03	3.18e-03	7.11e-03
16	6.52e-03	5.38e-04	5.24e-04	1.85e-04
32	1.69e-03	1.36e-04	1.32e-04	4.74e-05
64	4.32e-04	3.43e-05	3.40e-05	1.19e-05
128	1.09e-04	8.60e-06	8.58e-06	3.01e-06

Example 5.6 Next, consider the Fredholm integro-differntial equation [22],

$$u''(t) = \exp(t) - t + t\int_0^1 s\,u(s)\,ds,\ u(0) = 1, u'(0) = 1, \quad 0 \le t \le 1 \tag{5.6}$$

which has the exact solution $u(t) = \exp(t)$.
Integrating twice Eq. (5.6) w.r.t t, we get Fredholm integral equation,

$$u(t) = \exp(t) - \frac{t^3}{6} + \frac{t^3}{6}\int_0^1 s\,u(s)\,ds,$$

Solving above equation using the proposed method, we get CAS wavelet coefficients and substituting these coefficients in $u(t) \cong Y^T\Psi(t)$. Maximum error analysis is compared with existing method is shown in table 6.

Table 6: Maximum error analysis of the example 5.6

N	$E_{\max}$(HW)	$E_{\max}$(CAS)
8	3.07e-03	4.09e-04
16	8.48e-04	1.13e-04
32	2.22e-04	2.96e-04
64	5.69e-05	7.60e-06
128	1.44e-05	1.92e-06

Example 5.7 Next, consider the Volterra integro-differential equation [22],

$$u'(t) = 1 + \int_0^t u(s)\,ds,\ u(0) = 0,\quad 0 \le t \le 1 \tag{5.7}$$

which has the exact solution $u(t) = \sinh(t)$.

Integrating Eq. (5.7) w.r.t t, we get Volterra integral equation,

$$u(t) = t + \int_0^t (t-s)\,u(s)\,ds,$$

We applied the CAS wavelet approach and solved Eq. (5.7) yields the approximate value of $u(t)$ with the help of CAS wavelet coefficients. Maximum error analysis is shown in table 7 are compared with existing methods.

Table 7: Maximum error analysis of the example 5.7

N	$E_{\max}$(HW)	$E_{\max}$(LW)	$E_{\max}$(BW)	$E_{\max}$(CAS)
8	1.13e-02	2.38e-03	4.05e-03	8.97e-04
16	3.03e-03	6.19e-04	6.92e-04	2.37e-04
32	7.85e-04	1.57e-04	1.52e-04	6.10e-05
64	1.99e-04	3.95e-05	3.92e-05	1.54e-05
128	5.04e-05	9.91e-06	9.89e-06	3.89e-06

Example 5.8 Next, consider the Volterra-Fredholm integral equation [23],

$$u(t) = \cos(t)\left(\frac{1}{2}t - \frac{1}{4}\right) + \frac{1}{4}\cos(2-t) + \int_0^t \sin(t-s)\,u(s)\,ds + \int_0^1 \cos(t-s)\,u(s)\,ds,\quad 0 \le t \le 1 \tag{5.8}$$

which has the exact solution $u(t) = \sin(t)$. Solving Eq. (5.8) using the proposed method, we get CAS wavelet coefficients. Maximum error analysis is shown in table 5 and compared with the existing methods.

Let us approximate $f(t)$, $u(t)$, $k_1(t,s)$ and $k_2(t,s)$ as given in Eq. (3.5), Eq. (3.9) and Eq. (3.14) using the collocation points, we get an system of N equations with N unknowns,

i.e., $(I - K_1 - K_2)Y = X$ where, I is an identity matrix,

we find, X = [-0.1177 -0.0474 -0.0330 -0.0468 0.1146 -0.0353 -0.0224 -0.0287],

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$$K_1 = \begin{bmatrix} 0.4903 & -0.0027 & 0.0000 & 0.0027 & 0.4303 & 0.0185 & 0.0147 & 0.0233 \\ -0.0027 & 0.0039 & 0.0027 & 0.0039 & -0.0233 & 0.0034 & 0.0023 & 0.0032 \\ 0.0000 & 0.0027 & 0.0019 & 0.0027 & -0.0147 & 0.0025 & 0.0017 & 0.0023 \\ 0.0027 & 0.0039 & 0.0027 & 0.0039 & -0.0185 & 0.0036 & 0.0025 & 0.0034 \\ 0.4303 & -0.0233 & -0.0147 & -0.0185 & 0.4903 & -0.0027 & 0.0000 & 0.0027 \\ 0.0185 & 0.0034 & 0.0025 & 0.0036 & -0.0027 & 0.0039 & 0.0027 & 0.0039 \\ 0.0147 & 0.0023 & 0.0017 & 0.0025 & 0.0000 & 0.0027 & 0.0019 & 0.0027 \\ 0.0233 & 0.0032 & 0.0023 & 0.0034 & 0.0027 & 0.0039 & 0.0027 & 0.0039 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0.0386 & 0.0271 & 0.0153 & 0.0164 & 0 & 0 & 0 & 0 \\ -0.0164 & -0.0155 & -0.0054 & -0.0078 & 0 & 0 & 0 & 0 \\ -0.0153 & -0.0053 & -0.0077 & -0.0054 & 0 & 0 & 0 & 0 \\ -0.0271 & -0.0073 & -0.0053 & -0.0155 & 0 & 0 & 0 & 0 \\ 0.2351 & 0.0369 & 0.0269 & 0.0395 & 0.0386 & 0.0271 & 0.0153 & 0.0164 \\ -0.0395 & 0.0019 & 0.0012 & 0.0014 & -0.0164 & -0.0155 & -0.0054 & -0.0078 \\ -0.0269 & 0.0015 & 0.0009 & 0.0012 & -0.0153 & -0.0053 & -0.0077 & -0.0054 \\ -0.0369 & 0.0023 & 0.0015 & 0.0019 & -0.0271 & -0.0073 & -0.0053 & -0.0155 \end{bmatrix}$$

By solving this system we obtain the Hermite wavelet coefficient,

Y = [0.1707 -0.0610 -0.0423 -0.0591 0.4735 -0.0480 -0.0319 -0.0426],

Then, substituting $u(t)\ ;\ Y^T\Psi(t)$, we get the approximate solution of Eq. (5.8) are shown in table 8. Maximum error analysis is shown in table 9.

Table 8: Numerical results of the example 5.8

t	Exact	CAS wavelet
0.0625	0.0625	0.0596
0.1875	0.1864	0.1831
0.3125	0.3074	0.3037
0.4375	0.4237	0.4195
0.5625	0.5333	0.5286
0.6875	0.6346	0.6295
0.8125	0.7260	0.7205
0.9375	0.8061	0.8001

Table 9: Maximum error analysis of the example 5.8

N	E_{max} (HW)	E_{max} (LW)	E_{max} (BW)	E_{max} (CAS)
8	2.66e-02	1.15e-02	5.82e-03	5.99e-03
16	6.62e-03	2.92e-03	2.60e-03	1.50e-03
32	1.66e-03	7.38e-04	7.38e-04	3.79e-04
64	4.19e-04	1.85e-04	1.85e-04	9.53e-05

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VI. CONCLUSION

In this paper, we proposed the CAS wavelet collocation method for the numerical solution of integral and integro-differential equations. Using the proposed method, these equations are reduced to the system of algebraic equations with unknown coefficients. Solving this system of equations we obtain the approximate solution with the help of Matlab. Numerical results are compared with exact solutions and existing methods as shown in tables. Error analysis shows the accuracy and effectiveness of the present scheme is approached through the illustrative examples.

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